Real Oscillators

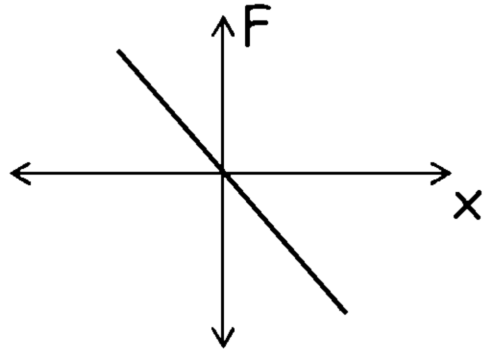
… constant forces $\rightarrow$ integrate EOM $\rightarrow$ parabolic trajectories.

… linear restoring force $\rightarrow$ guess EOM solution $\rightarrow$ SHM

… nonlinear restoring forces $\rightarrow$?

linear spring

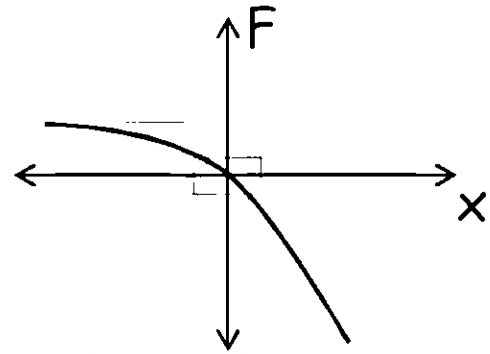
$$F = -kx$$



$$-kx = m\ddot{x}$$

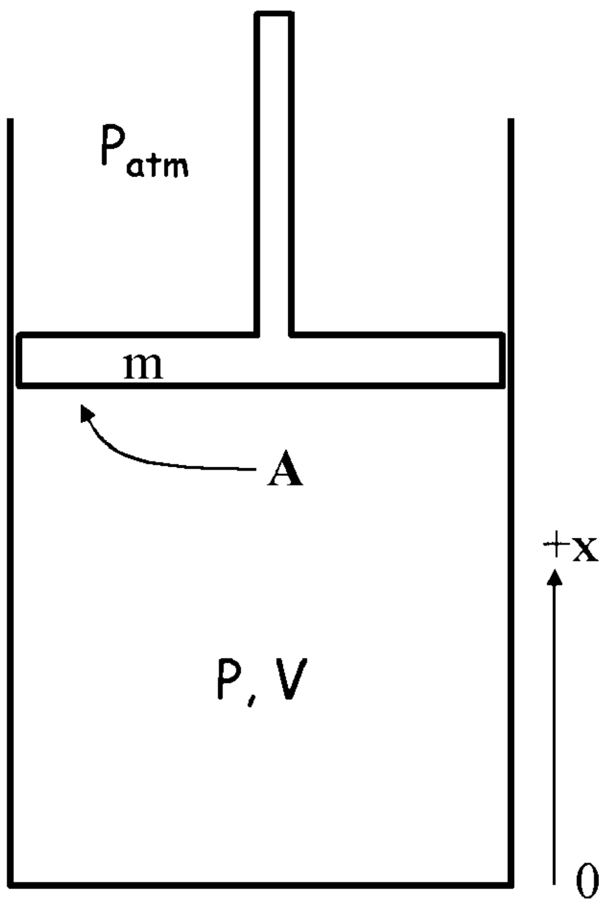
nonlinear spring?

$$F = c\left(1 - e^{x}\right)$$



$$c\left(1 - e^{x}\right) = m\ddot{x}$$

The spring of air :



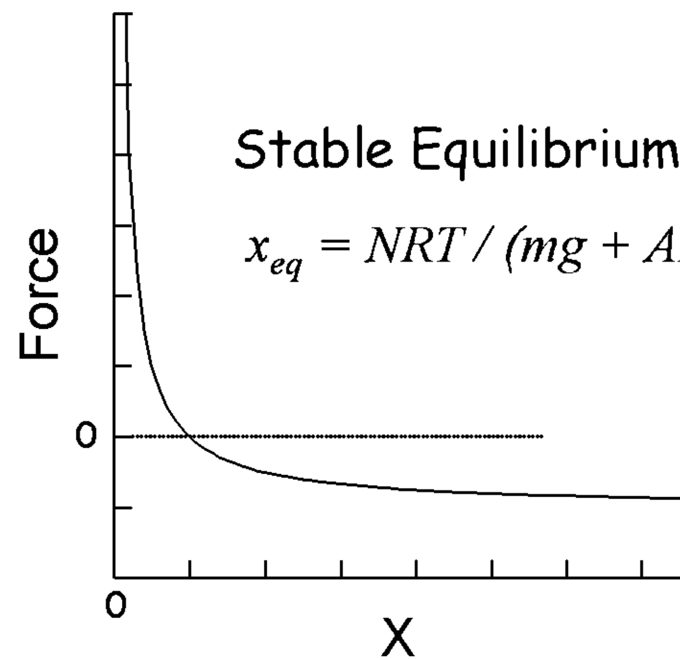
$$\Sigma F = -mg - AP_{atm} + AP = m\ddot{x}$$

use Ideal Gas Law: $PV = NRT$

$$-mg - AP_{atm} + A\frac{NRT}{V} = m\ddot{x}$$

chamber volume: $V = Ax$

$$\boxed{-mg - AP_{atm} + \frac{NRT}{x} = m\ddot{x}}$$ EOM



Stable Equilibrium at

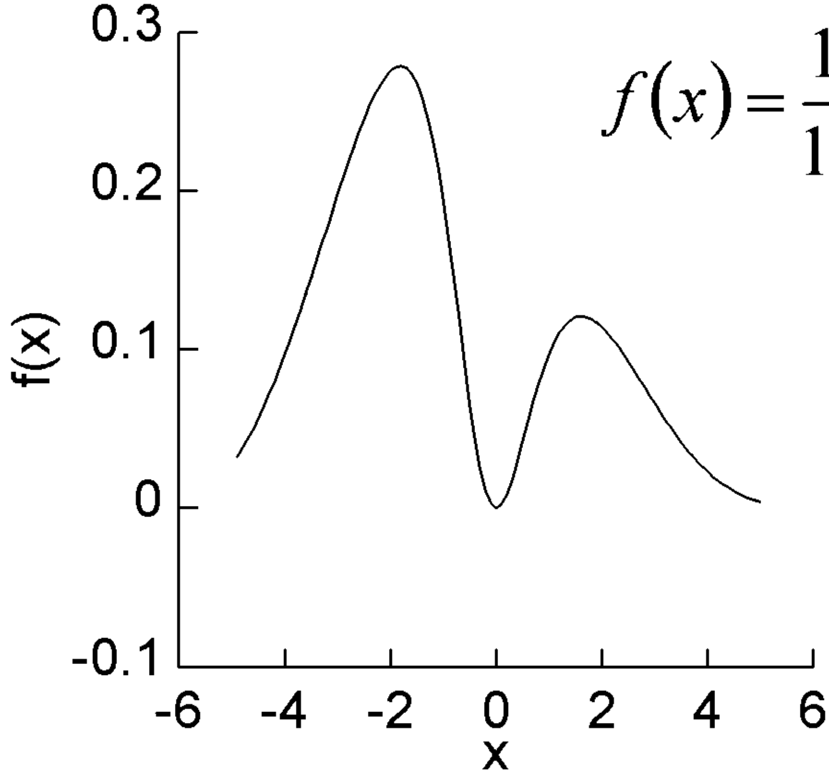
$x_{eq} = NRT / (mg + AP_{atm})$

Taylor Series Expansions:

$$f(x)=\sum_{n=0}^{\infty}\frac{\frac{d^n f}{dx^n}(a)}{n!}(x-a)^n$$

Turns a function into a polynomial near $x = a$

Example:



$$f(x)=\frac{1-\cos(x)}{1+x^2+e^x}$$

Expand NRT/x around x_{eq}:

$$\left[-mg - AP_{atm} + \frac{NRT}{x_{eq}} - \frac{NRT}{x_{eq}^{\;2}}\left(x - x_{eq}\right) + \frac{NRT}{x_{eq}^{\;3}}\left(x - x_{eq}\right)^2 - \ldots\right] = m\ddot{x}$$

$$\left[0 - \frac{NRT}{x_{eq}^{\;2}}\left(x - x_{eq}\right) + \frac{NRT}{x_{eq}^{\;3}}\left(x - x_{eq}\right)^2 - \ldots\right] = m\ddot{x}$$

Is it safe to linearize it? Better check a unitless ratio. How about:

$$\left(\frac{x - x_{eq}}{x_{eq}}\right)$$

(Yes, excellent choice Dr. Hafner!)

$$\frac{NRT}{x_{eq}}\left[0-\left(\frac{x-x_{eq}}{x_{eq}}\right)+\left(\frac{x-x_{eq}}{x_{eq}}\right)^2-\ldots\right]=m\ddot{x}$$

Displacement 5% of x_{eq}: 0 .05 .0025 ....

$$-\frac{NRT}{x_{eq}^{\ 2}}\left(x-x_{eq}\right)\approx m\ddot{x}$$

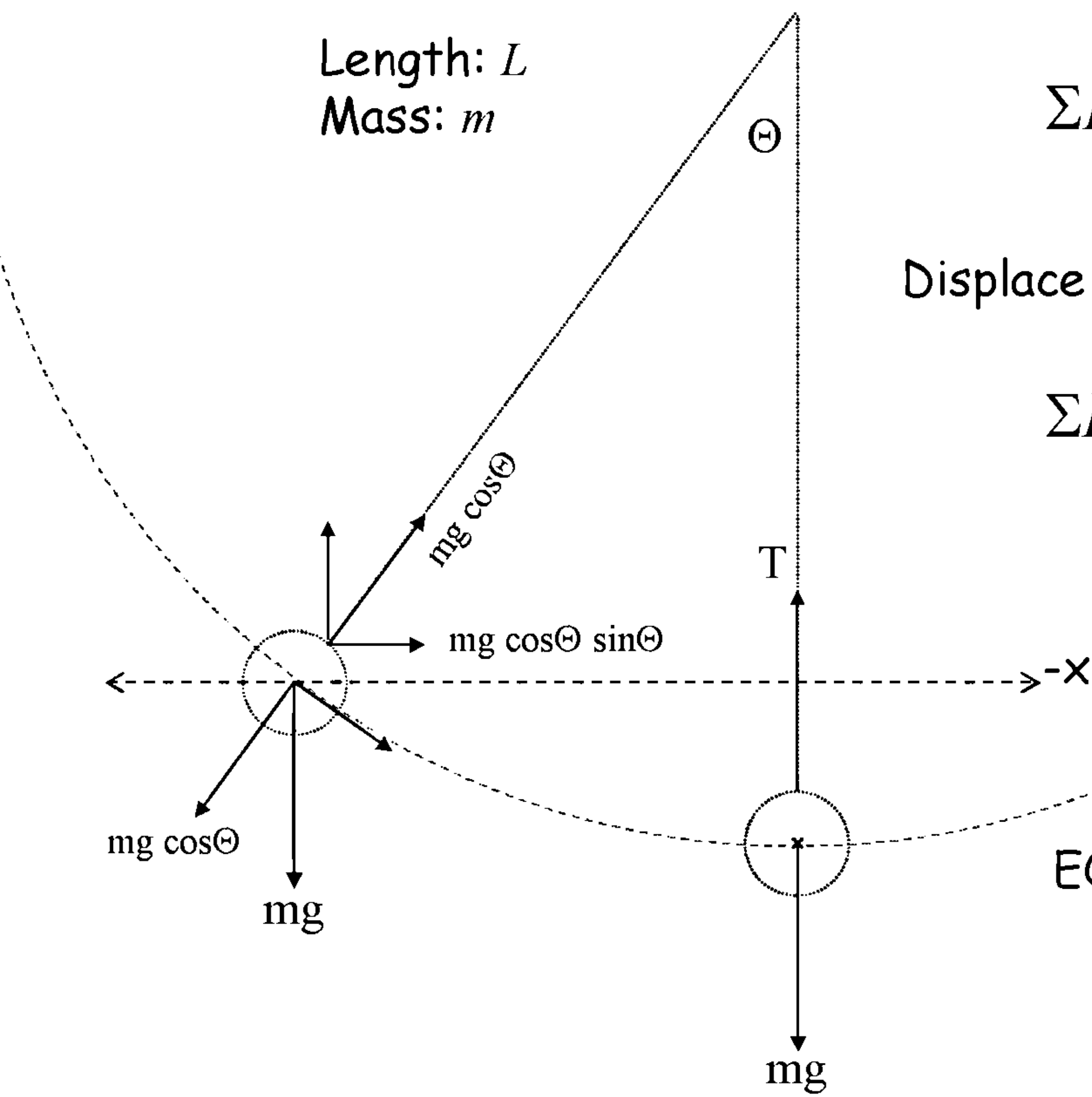
Perhaps you would prefer....

$$-\frac{NRT}{mx_{eq}^{\ 2}}\left(x-x_{eq}\right)\approx \ddot{\left(x-x_{eq}\right)}$$

SHM with $\omega_o=\frac{\sqrt{NRT/m}}{x_{eq}}$

Simple Pendulum:

Length: L
Mass: m



Stable Equilibrium:

$$\Sigma F_x = 0 \qquad \Sigma F_y = T - mg = 0$$

Displace by Θ:

$$\Sigma F_x = -mg\cos(\Theta)\sin(\Theta)$$

$$= -mg\frac{\sqrt{L^2 - x^2}}{L}\frac{x}{L}$$

EOM: $-mg\dfrac{\sqrt{L^2 - x^2}}{L}\dfrac{x}{L} = m\ddot{x}$

Expand it!

$$-\frac{g}{L^2}x\sqrt{L^2-x^2}=\ddot{x}$$

Derivatives:

$$f=x\sqrt{L^2-x^2}$$

$$f'=\sqrt{L^2-x^2}-x^2\left(L^2-x^2\right)^{-\frac{1}{2}}$$

$$f''=-3x\left(L^2-x^2\right)^{-\frac{1}{2}}-x^3\left(L^2-x^2\right)^{-\frac{3}{2}}$$

$$f'''=-3\left(L^2-x^2\right)^{-\frac{1}{2}}-6x^2\left(L^2-x^2\right)^{-\frac{3}{2}}-3x^4\left(L^2-x^2\right)^{-\frac{5}{2}}$$

$$-\frac{g}{L^2}\left[0 + Lx + 0 - \frac{3}{6L}x^3 \ldots\right] = \ddot{x}$$

Now express as a unitless ratio of the dependent variable and some parameter of the system:

$$-g\left[0 + \left(\frac{x}{L}\right) + 0 - \frac{1}{2}\left(\frac{x}{L}\right)^3 \ldots\right] = \ddot{x}$$

Displacement 5% of length: 0 .05 0 .0000625 …

$$-\frac{g}{L}x \approx \ddot{x}$$

SHM with $\omega_o = \sqrt{\frac{g}{L}}$

The world is not linear. However, one can use a Taylor expansion to linearize an EOM by assuming only small perturbations around a point of stable equilibrium (which may not be the origin).