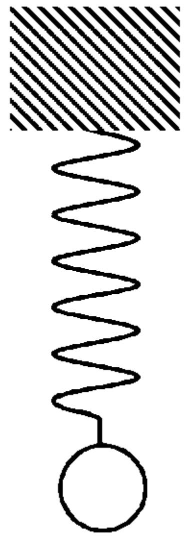
Superposition

Free SHM



EOM: $-kx = m\ddot{x}$

Solutions: $x_1(t) = A_1 \cos(\omega_1 t + \phi_1)$

$$x_2(t) = A_2 \cos(\omega_2 t + \phi_2)$$

Superposition: the sum of solutions to an EOM is also a solution . . .
. . . *if the EOM is linear.*

x and its derivatives appear only to first power.

…or a linear combination:
$c_1x_1 + c_2x_2$

Is the linear combination useful?

initial displacement, begin at rest

$x_o,\ v_o = 0$

$$x_1(0) = A_1\cos(\phi_1) = x_0$$

$$\dot{x}_1(0) = -A_1\omega_o\sin(\phi_1) = 0$$

$$\phi_1 = 0, \quad x_0 = A_1$$

$$\boxed{x_1(t) = x_o\cos(\omega_o t)}$$

initial velocity, begin at origin

$v_o,\ x_o = 0$

$$x_2(0) = A_2\cos(\phi_2) = 0$$

$$\phi_2 = \tfrac{\pi}{2}$$

$$\dot{x}_2(0) = -A_2\omega_o\sin\left(\tfrac{\pi}{2}\right) = v_o$$

$$A_2 = \frac{-v_o}{\omega_o}$$

$$\boxed{x_2(t) = \frac{-v_o}{\omega_o}\cos\left(\omega_o t + \tfrac{\pi}{2}\right)}$$

initial displacement x_o *and* velocity v_o

$$x_3(0) = A_3 \cos(\phi_3) = x_0$$

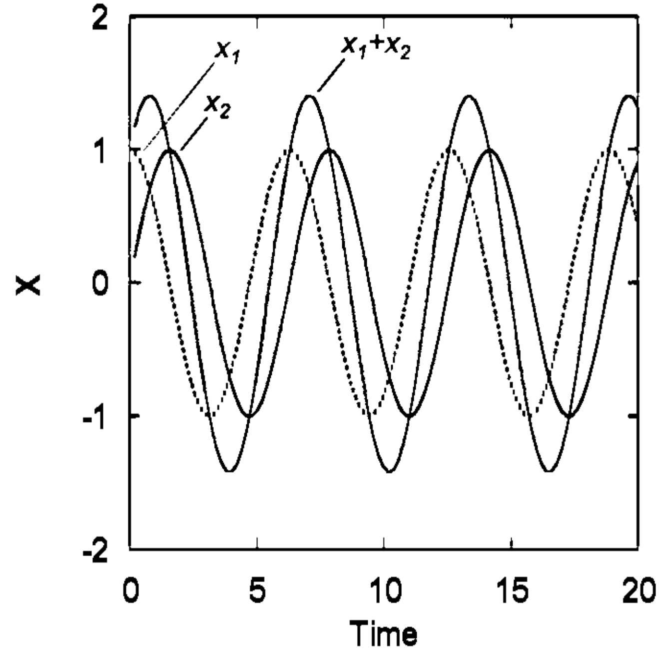
$$\dot{x}_3(0) = -A_3\omega_o \sin(\phi_3) = v_o$$

Solve each for A_3, equate, solve for ϕ:

$$\phi_3 = \tan^{-1}\left[\frac{-v_o}{\omega_o x_o}\right]$$

Plug back into top equation to get A_3:

$$A_3 = \frac{x_o}{\cos\left(\tan^{-1}(-v_o/\omega_o x_o)\right)}$$



Superposition: The motion resulting from two simultaneous initial conditions is equal to the sum of the motions resulting from each initial condition. . . . . . *if the EOM is linear.*

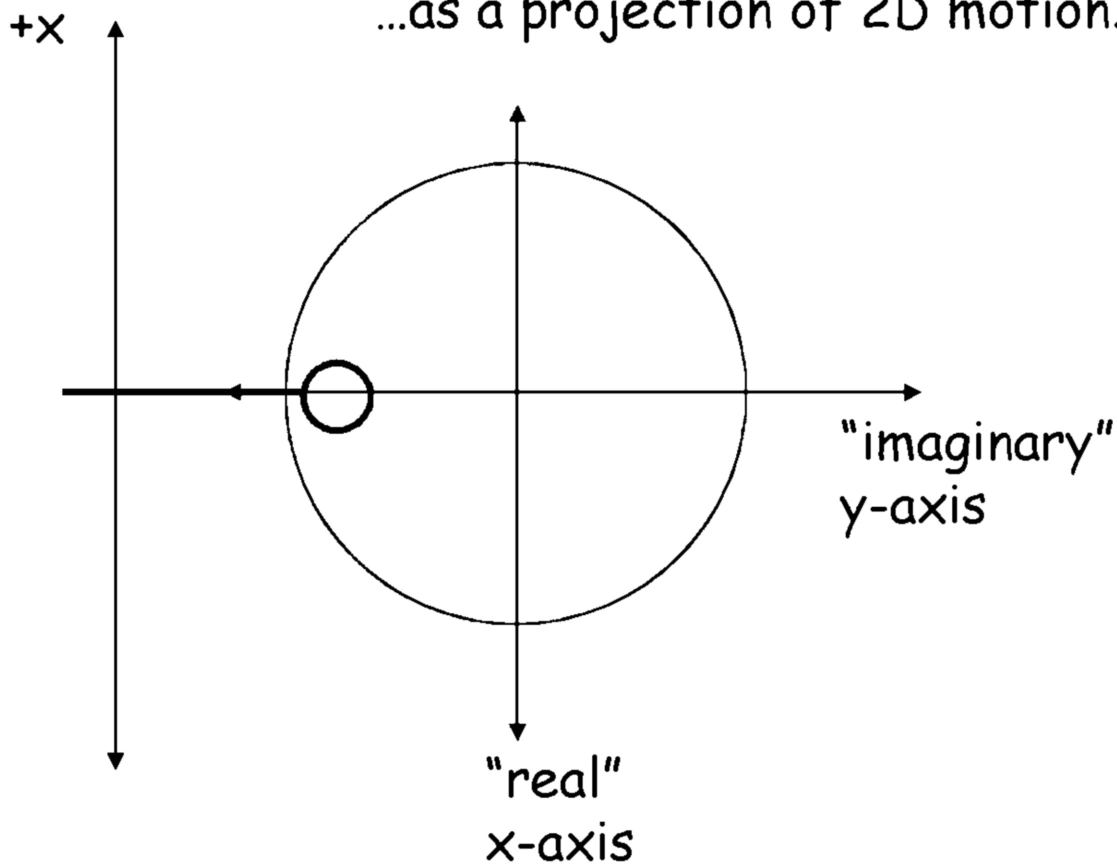
Solution:

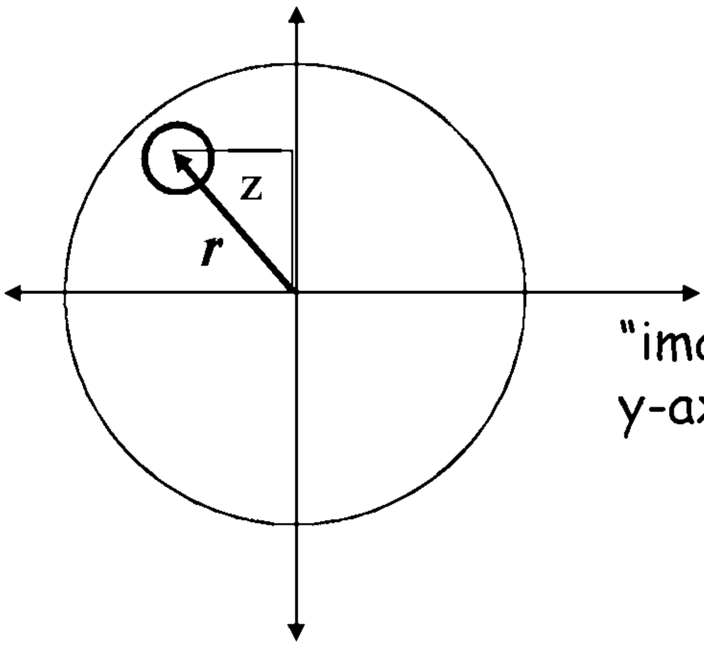
$$x_3(t) = \frac{x_o}{\cos\left(\tan^{-1}(-v_o/\omega_o x_o)\right)} \cos\left(\omega_o t + \tan^{-1}(-v_o/\omega_o x_o)\right)$$

The trig is getting complicated, let's try something else…

Imagine SHM in 1D…

…as a projection of 2D motion.





Describe the position…

…geometrically OR …algebraically

$\vec{r} = a\hat{i} + b\hat{j}$

$z = a + jb$

j -> "rotate 90 degrees"

$j^2 b$ -> "go along original axis in opposite direction"

$$j^2 = -1$$

$$j = \sqrt{-1}$$ *… an imaginary number!*

$$\sin(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \frac{t^9}{9!} \cdots \qquad \cos(t) = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!} \cdots$$

Circular motion in complex plane:

$$\cos(t) + j\sin(t) = 1 + jt - \frac{t^2}{2!} - j\frac{t^3}{3!} + \frac{t^4}{4!} + j\frac{t^5}{5!} \cdots$$

$$= 1 + jt + \frac{(jt)^2}{2!} + \frac{(jt)^3}{3!} + \frac{(jt)^4}{4!} + \frac{(jt)^5}{5!} \cdots$$

...expansion of *exp(jt)*!

$$\cos(t) + j\sin(t) = e^{jt}$$

"Euler's Formula" - imaginary exponentials oscillate !

Describe SHM in the complex plane…

$$z(t) = Ae^{j(\omega t+\phi)}$$

Does it work?

$$-k\left[Ae^{j(\omega t+\phi)}\right] = m\frac{d^2}{dt^2}\left[Ae^{j(\omega t+\phi)}\right]$$

$$-kAe^{j(\omega t+\phi)} = -mA\omega^2 e^{j(\omega t+\phi)}$$

Yes, if: $k = m\omega^2$

A and ϕ take any value

Don't forget!!! $x(t) = \mathrm{Re}\left[z(t)\right] = A\cos(\omega t + \phi)$ *Keepin' it real !*

Same A and ω, but different ϕ:

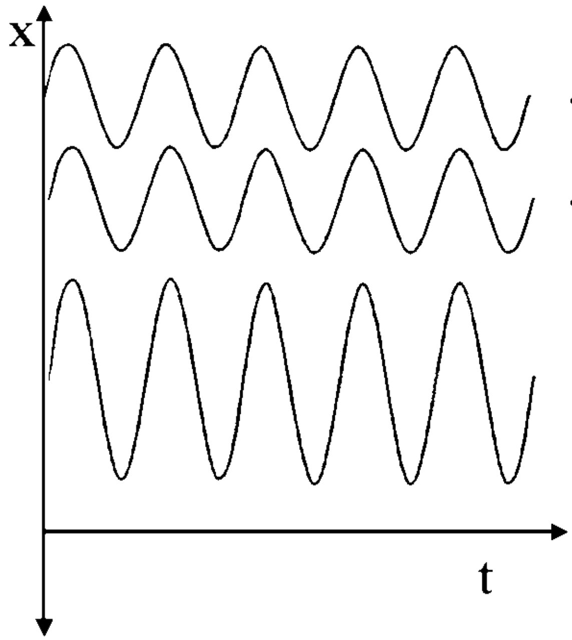
$$z_1 = Ae^{j(\omega t+\phi_1)} \qquad z_2 = Ae^{j(\omega t+\phi_2)}$$

$$z_{1+2} = A\left(e^{j(\omega t+\phi_1)} + e^{j(\omega t+\phi_2)}\right)$$

$$z_{1+2} = Ae^{j\omega t}\left(e^{j\phi_1} + e^{j\phi_2}\right) \quad \frac{e^{j\phi_1}}{e^{j\phi_1}}$$

$$z_{1+2} = Ae^{j(\omega t+\phi_1)}\left(1 + e^{j(\phi_2-\phi_1)}\right)$$

x_1 and x_2 in phase

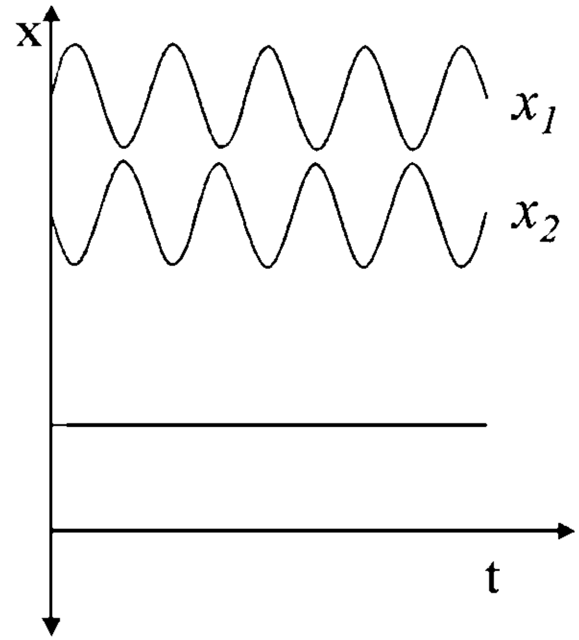


$$\phi_1 - \phi_2 = 0$$

$$z_{1+2} = 2Ae^{j(\omega t + \phi_1)}$$

$$x_{1+2} = 2A\cos(\omega t + \phi_1)$$

x_1 and x_2 out of phase



$$\phi_1 - \phi_2 = \pi$$

$$z_{1+2} = Ae^{j(\omega t + \phi_1)}\left(1 + e^{j\pi}\right)$$

$$z_{1+2} = 0$$

$$x_{1+2} = 0$$

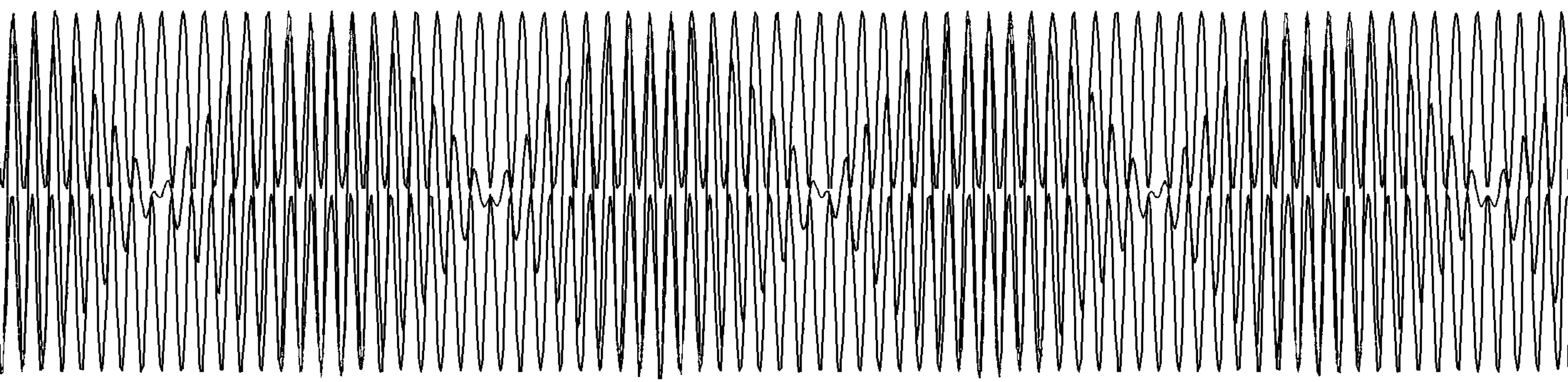
Same A and ϕ, different ω:

Stick with trig... $x_{1+2} = A[\cos(\omega_1 t) + \cos(\omega_2 t)]$

$$\cos(\alpha) + \cos(\beta) = 2\cos\left(\tfrac{\alpha+\beta}{2}\right)\cos\left(\tfrac{\alpha-\beta}{2}\right)$$

Then: $x_{1+2} = 2A\cos\left(\dfrac{\omega_1 + \omega_2}{2}t\right)\cos\left(\dfrac{\omega_1 - \omega_2}{2}t\right)$

$\omega_1 \approx \omega_2$ yields <u>beats</u>.



Systems with linear EOMs obey the Principle of Superposition: solutions to the EOM can be summed to make more complicated solutions.