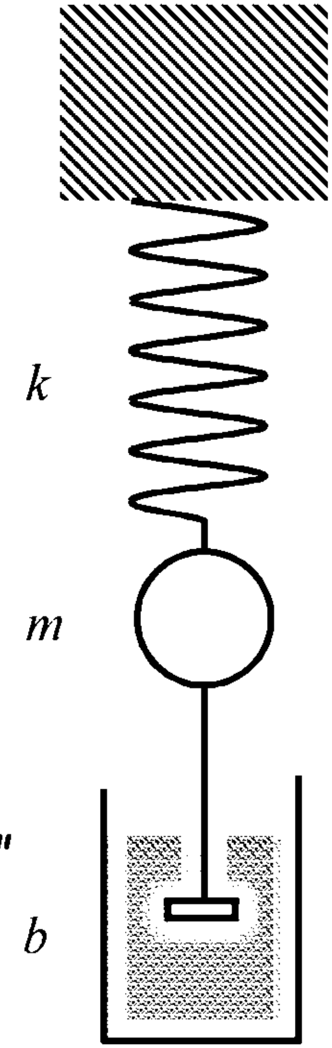
Damped SHM



"Damping Constant"

(kg/s)

$$\sum F = -b\dot{x} - kx = m\ddot{x}$$

$$\omega_o = \sqrt{k/m}$$

"Natural Frequency"

(rad/s)

$$\gamma = b/m$$

"Damping Parameter"

(s^{-1})

$$\boxed{\ddot{x} + \gamma\,\dot{x} + \omega_o^{\,2} x = 0}$$

EOM: damped oscillator

Guess a complex solution: $z(t) = Ae^{j(pt+\phi)}$

$$j^2 p^2 A e^{j(pt+\phi)} + \gamma j p A e^{j(pt+\phi)} + \omega_o{}^2 A e^{j(pt+\phi)} = 0$$

$$Ae^{j(pt+\phi)}\left(-p^2 + \gamma j p + \omega_o{}^2\right) = 0$$

$$A = 0$$

"trivial solution"

$$-p^2 + \gamma j p + \omega_o{}^2 = 0$$

Actually 2 equations:

<u>Real = 0</u>

$$-p^2 + \omega_o{}^2 = 0$$

$$p = \omega_o$$

<u>Imaginary = 0</u>

$$\gamma j p = 0$$

$$\gamma = 0$$

... also trivial !

Try a complex frequency: $z(t) = Ae^{j((n+js)t+\phi)}$

$$-(n+js)^2 + \gamma\, j(n+js) + \omega_o{}^2 = 0$$

$$-n^2 - 2jns + s^2 + \gamma jn - \gamma s + \omega_o{}^2 = 0$$

Real

$$-n^2 + s^2 - \gamma s + \omega_o{}^2 = 0$$

$$-n^2 + \frac{\gamma^2}{4} - \frac{\gamma^2}{2} + \omega_o{}^2 = 0$$

$$\boxed{n^2 = \omega_o{}^2 - \frac{\gamma^2}{4}}$$

Imaginary

$$-2jns + \gamma jn = 0$$

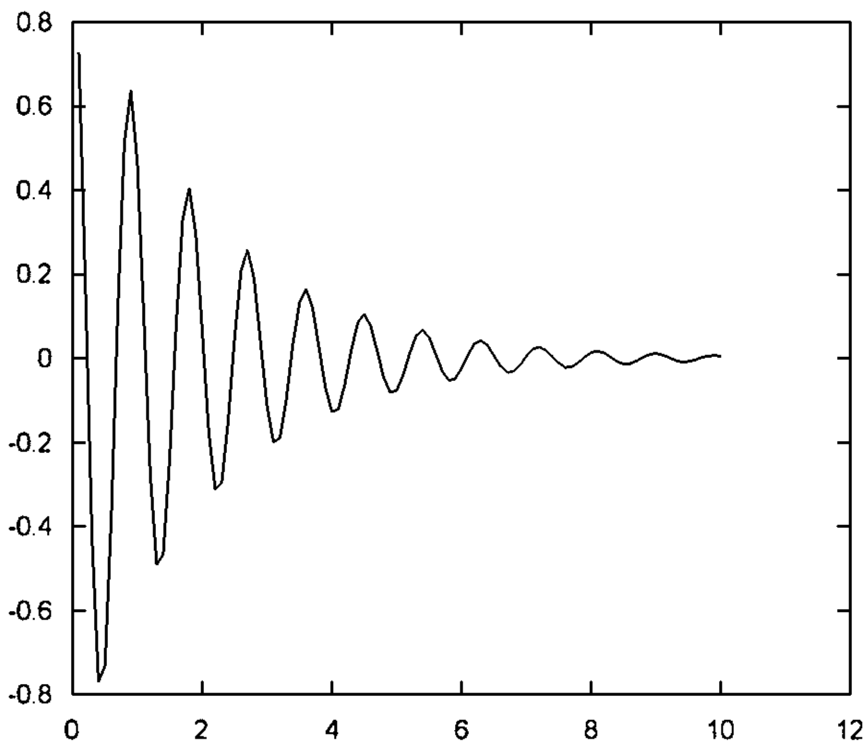
$$\boxed{s = \frac{\gamma}{2}}$$

A, ϕ are free constants.

$$z(t) = Ae^{j\left(\left(\sqrt{\omega_o{}^2 - \frac{\gamma^2}{4}} + j\frac{\gamma}{2}\right)t + \phi\right)}$$

$$z(t) = Ae^{-\frac{\gamma}{2}t} e^{j\left(\sqrt{\omega_o{}^2 - \frac{\gamma^2}{4}}\, t + \phi\right)}$$

$$x(t) = Ae^{-\frac{\gamma}{2}t} \cos\left(\sqrt{\omega_o{}^2 - \frac{\gamma^2}{4}}\, t + \phi\right)$$



* amplitude *decays* due to damping

* frequency *reduced* due to damping

How damped?

Quality factor: unitless ratio of natural frequency to damping parameter

$$Q \equiv \frac{\omega_0}{\gamma}$$

Sometimes write solution in terms of wo and Q

$$z(t) = Ae^{-\frac{\omega_o}{2Q}t} e^{j\left(\omega_o \sqrt{1-\frac{1}{4Q^2}}\, t+\phi\right)}$$

Sometimes write EOM in terms of wo and Q:

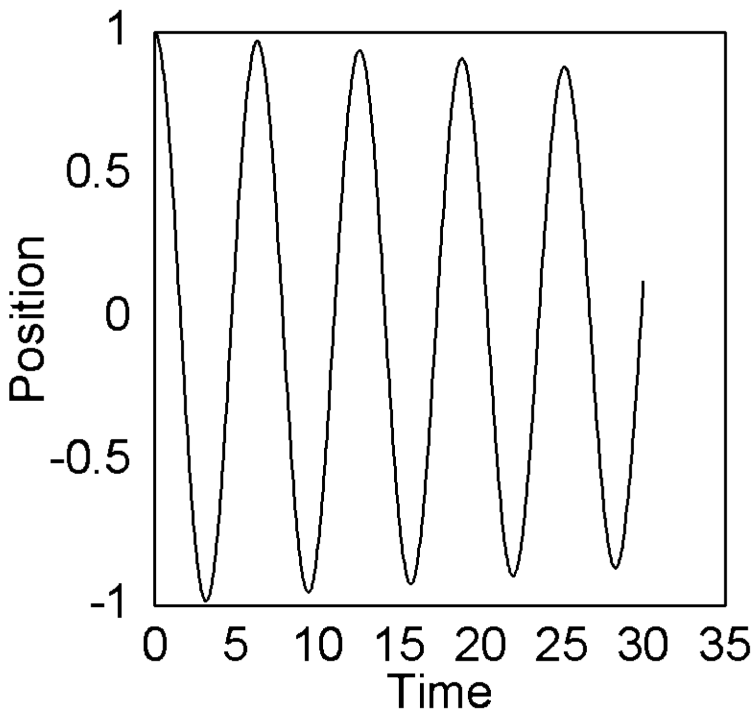
$$\ddot{x} + \frac{\omega_o}{Q}\dot{x} + \omega_o{}^2 x = 0$$

1. "Under Damped" or "Lightly Damped": $Q >> 1$

Oscillates at $\sim\omega_o$ (slightly less)

Looks like SHM (constant A) over a few cycles:

$\omega_o = 1,\ \gamma = .01,\ Q = 100,\ x_o = 1$



Amplitude drops by 1/e in Q/π cycles.

2. "Over Damped": $Q << 1 \quad \omega_o << \gamma$

$$z(t) = Ae^{-\frac{\gamma}{2}t}e^{j\left(\underbrace{\sqrt{\omega_o{}^2-\frac{\gamma^2}{4}}}_{\text{imaginary!}}\,t+\phi\right)}$$

$$z(t) = Ae^{-\frac{\gamma}{2}t}e^{j\left(j\sqrt{\frac{\gamma^2}{4}-\omega_o{}^2}\,t+\phi\right)}$$

$$z(t) = Ae^{-\frac{\gamma}{2}t}e^{-\sqrt{\frac{\gamma^2}{4}-\omega_o{}^2}\,t}\cancel{e^{j\phi}}$$

part of A

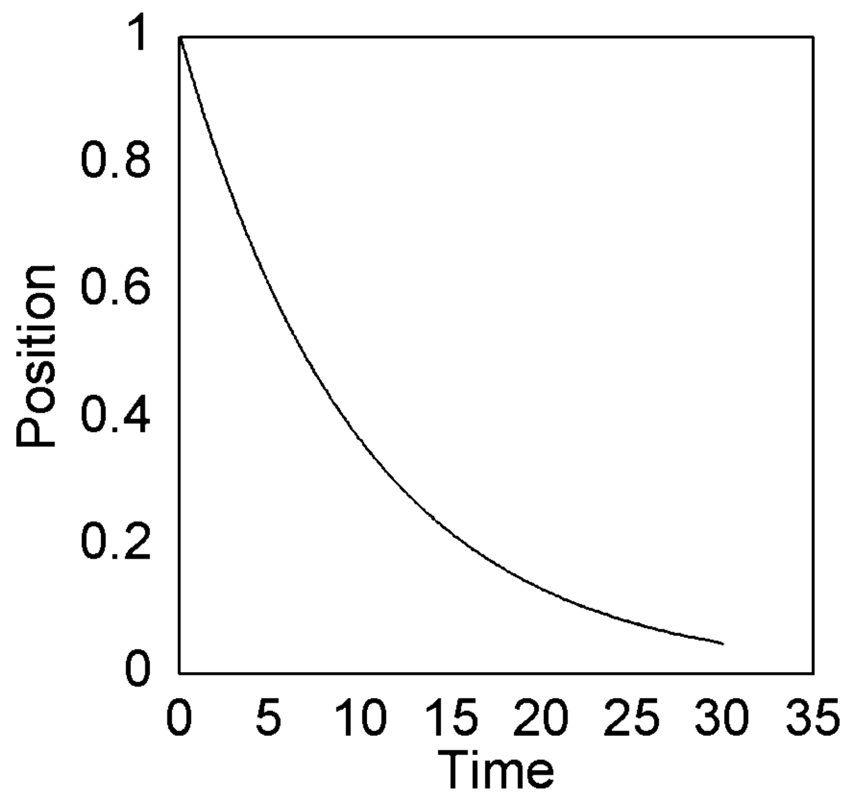
Still need two constants for the 2nd order EOM:

$$z(t) = A_1e^{-\left(\frac{\gamma}{2}+\sqrt{\frac{\gamma^2}{4}-\omega_o{}^2}\right)t} + A_2e^{-\left(\frac{\gamma}{2}-\sqrt{\frac{\gamma^2}{4}-\omega_o{}^2}\right)t}$$

No oscillations!

Over Damped

$\omega_o = 1,\ \gamma = 10,\ Q = .1,\ x_o = 1$



3 "Critically Damped": $Q = 0.5$ $\gamma = 2\omega_o$

$$z(t) = Ae^{-\frac{\gamma}{2}t}e^{j\left(\cancelto{0}{\sqrt{\omega_o^2-\frac{\gamma^2}{4}}}\,t+\phi\right)}$$

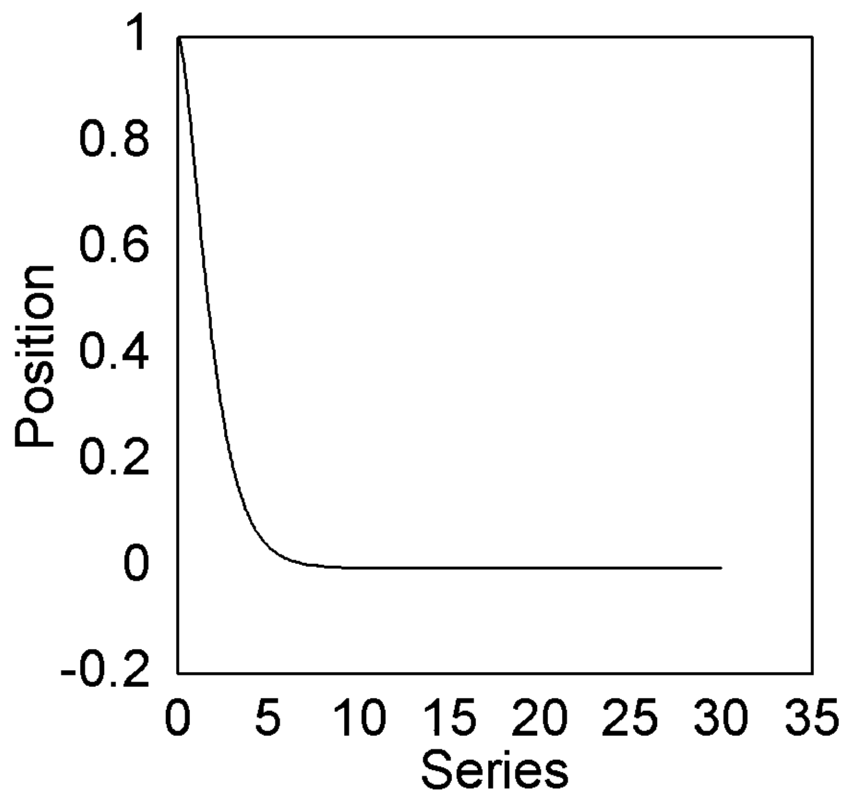
$$z(t) = \left(A_1 + A_2\right)e^{-\left(\frac{\gamma}{2}\right)t}$$

…really just one constant, and we need two. Real solution:

$$z(t) = \left(A + Bt\right)e^{-\left(\frac{\gamma}{2}\right)t}$$

Critically Damped

$\omega_o = 1, \gamma = 2, Q = .5, x_o = 1$



Fastest approach to zero with no overshoot.

Real oscillators lose energy due to damping. This can be represented by a damping force in the equation of motion, which leads to a decaying oscillation solution. The relative size of the resonant frequency and damping parameter define different behaviors: lightly damped, critically damped, or over damped.