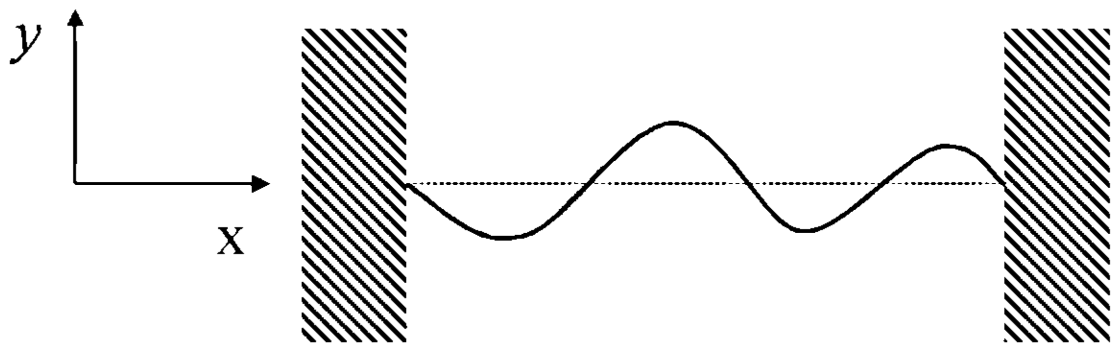
Continua

Continuum Archetype: The Stretched String



length = L
mass/length = μ
tension = T

Consider displacement (y) along position (x) at any time (t): $y(x,t)$

Two independent variables!!!

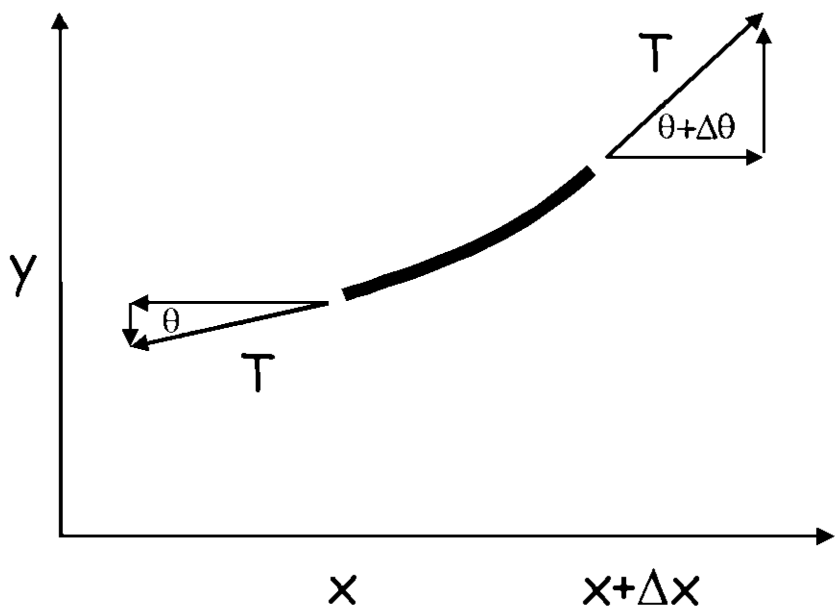
Derivative of $y(t)$

$$\frac{dy}{dt} = \lim_{\Delta t \to 0} \frac{y(t+\Delta t) - y(t)}{\Delta t}$$

Partial Derivative of $y(x,t)$

$$\frac{\partial y}{\partial t} = \lim_{\Delta t \to 0} \frac{y(x, t+\Delta t) - y(x,t)}{\Delta t}$$

small offset and curved segment:



$$\sum F_x = T\cos(\theta+\Delta\theta) - T\cos(\theta)$$

$$\sum F_y = T\sin(\theta+\Delta\theta) - T\sin(\theta)$$

Linearize for small displacements and therefore small angles:

$$\sum F_x \approx 1 - 1 \approx 0$$

No force in the x-direction

$$\sum F_y \approx T\times(\theta+\Delta\theta) - T\times(\theta)$$

Small angle θ (in radians) =
opp/hyp = opp/adj = tan(θ)=
rise/run = slope = dy/dx

Switch from θs to partial derivatives (slopes):

$$\sum F_y \approx T \times \frac{\partial y}{\partial x}(x+\Delta x) - T \times \frac{\partial y}{\partial x}(x)$$

Multiply by $\Delta x / \Delta x$:

$$\sum F_y \approx T\,\Delta x \left[\frac{\frac{\partial y}{\partial x}(x+\Delta x) - \frac{\partial y}{\partial x}(x)}{\Delta x} \right]$$

Limit as $\Delta \mathrm{x} \to 0$:

$$\sum F_y \approx T\,\Delta x \frac{\partial^2 y}{\partial x^2}$$

So the vertical string force is proportional to *curvature.*

$$T \, \Delta x \frac{\partial^2 y}{\partial x^2} = m a_y$$

$$T \Delta x \frac{\partial^2 y}{\partial x^2} = \mu \, \Delta x \, \frac{\partial^2 y}{\partial t^2}$$

$$\frac{\partial^2 y}{\partial x^2} = \left(\frac{\mu}{T}\right) \frac{\partial^2 y}{\partial t^2}$$ define: $$v = \sqrt{\frac{T}{\mu}}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$$

WAVE EQUATION

Any function: $y = f(x - vt)$ or $y = f(x + vt)$ is a solution to the wave equation!

chain rule:

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial (x - vt)} \frac{\partial (x - vt)}{\partial x}$$

$$\frac{\partial y}{\partial t} = \frac{\partial y}{\partial (x - vt)} \frac{\partial (x - vt)}{\partial t}$$

$$\frac{\partial y}{\partial x} = \frac{\partial y}{\partial (x - vt)} 1$$

$$\frac{\partial y}{\partial t} = -v \frac{\partial y}{\partial (x - vt)}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial \frac{\partial y}{\partial (x - vt)}}{\partial (x - vt)} \frac{\partial (x - vt)}{\partial x}$$

$$\frac{\partial^2 y}{\partial t^2} = -v \frac{\partial \frac{\partial y}{\partial (x - vt)}}{\partial (x - vt)} \frac{\partial (x - vt)}{\partial t}$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial (x - vt)^2}$$

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial (x - vt)^2}$$

We have not used the wave equation, we have only taken partial derivatives!

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial (x - vt)^2} \qquad \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial (x - vt)^2}$$

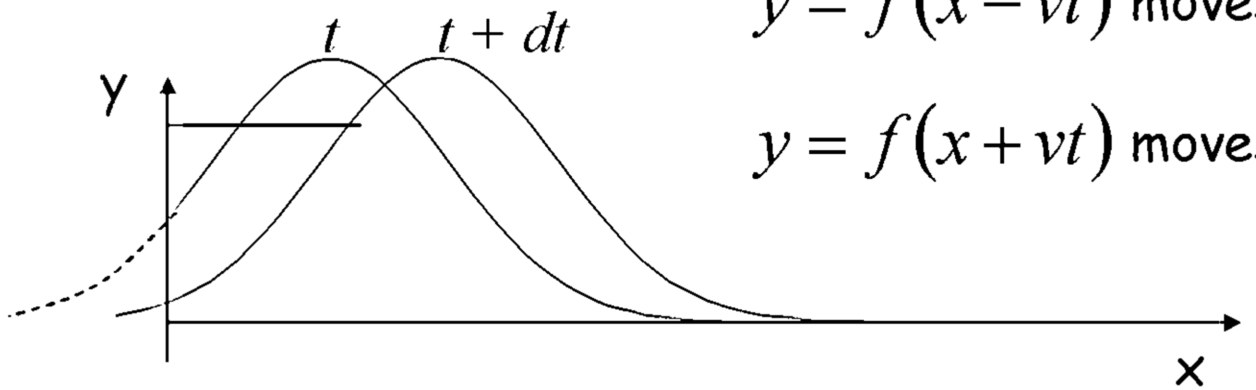
$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

Any function:

$$y = f(x - vt) \text{ or } y = f(x + vt)$$

is a "traveling wave" solution to the wave equation

For example, Gaussian: $y = Ae^{-(x-vt)^2/c^2}$



$y = f(x - vt)$ moves in +x

$y = f(x + vt)$ moves in -x

How fast? $y = f(x,t) = f(x + dx, t + dt)$

$$x - vt = (x + dx) - v(t + dt)$$

$$0 = dx - vdt$$

$$dx/dt = v$$