## Continua Continued

More Useful: find Normal Mode solutions:

$$y(x,t) = f(x)\cos(\omega t)$$

$$\frac{\partial^2 f}{\partial x^2} \cos(\omega t) = -\frac{1}{v^2} f(x) \omega^2 \cos(\omega t)$$

good solution if... 
$$\frac{d^2f}{dx^2} = -\frac{\omega^2}{v^2}f(x)$$

therefore if... 
$$f(x) = A \sin\left(\frac{\omega}{v}x + \phi\right)$$
 ... normal modes have a sinusoidal shape.

sinusoidal shape.

$$y(x,t) = A\sin\left(\frac{\omega}{v}x + \phi\right)\cos(\omega t)$$

Set  $\phi = \pi/2$  for the moment and apply product-to-sum:

$$y(x,t) = A\cos\left(\frac{\omega}{v}x\right)\cos(\omega t)$$

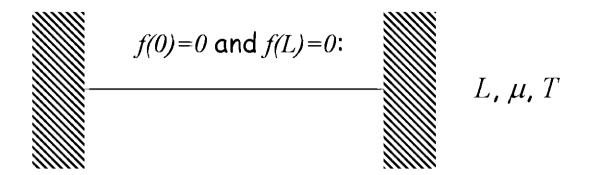
$$y(x,t) = \frac{A}{2} \left[ \cos \left( \frac{\omega}{v} x - \omega t \right) + \cos \left( \frac{\omega}{v} x + \omega t \right) \right]$$

Rearrange to f(x-vt)

$$y(x,t) = \frac{A}{2} \left[ \cos \left( \frac{\omega}{v} (x - vt) \right) + \cos \left( \frac{\omega}{v} (x + vt) \right) \right]$$

Normal mode = two counter-propagating traveling waves

Infinite strings give infinite solutions. Try a bound string:



Wave equation still applies.

Normal mode solutions still apply.

But now we can apply boundary conditions

$$f(0) = A\sin(\phi) = 0$$

$$f(L) = A\sin\left(\frac{\omega}{v}L\right) = 0$$

$$\phi = 0$$

$$\frac{\omega L}{v} = n\pi \qquad n = 1, 2, 3, 4, ...$$

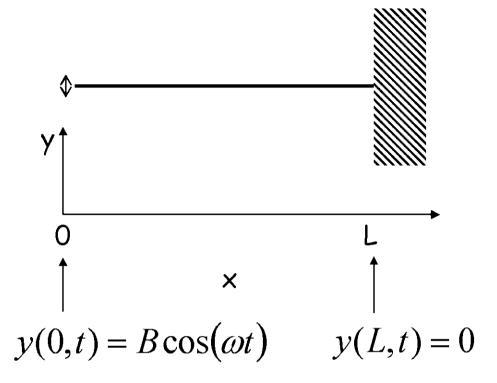
Boundary conditions reveal specific normal mode frequencies! (for the clamped string)

$$y_n(x,t) = A_n \sin\left(\frac{\omega_n}{v}x\right) \cos(\omega_n t)$$
  $\omega_n = \frac{n\pi}{L} \sqrt{\frac{T}{\mu}}$ 

These are the normal mode solutions, NOT the solution to some driving force or initial condition!!

If we shake one end....

With these Boundary conditions:



$$y(x,t) = f(x)\cos(\omega t)$$
 ... is this a good solution?

$$f(x) = A\sin\left(\frac{\omega}{v}x + \phi\right)$$

Boundary conditions reveal normal mode wavelengths & frequencies! (for the clamped string)

Apply boundary condition:

$$y(L,t) = A \sin\left(\frac{\omega}{v}L + \phi\right)\cos(\omega t) = 0$$

Sine is zero if:

$$\frac{\omega}{v}L + \phi = n\pi$$
  $n = 1, 2, 3, 4, ...$ 

$$\phi = n\pi - \frac{\omega}{v}L$$

Apply other boundary condition:

$$y(0,t) = A\sin(\phi)\cos(\omega t) = B\cos(\omega t)$$

$$A = \frac{B}{\sin(\phi)}$$

Wavenumber: 
$$k = \frac{2\pi}{\lambda}$$

French is the only book still in print that defines  $k = 1/\lambda$  !!!

The equation of motion for a stretched string (a continuum) is the wave equation. It has traveling wave and normal mode solutions (which are really the same thing).