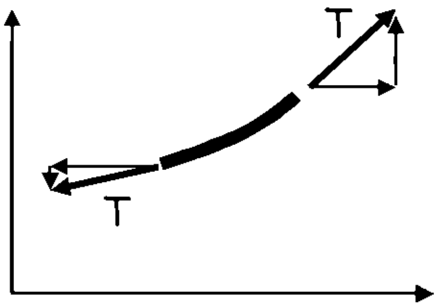
Stretched string



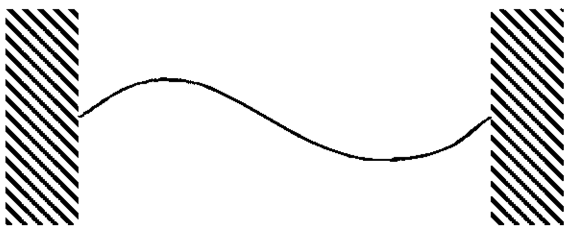

Newton's 2nd

$$\sum F = ma$$

Wave Equation

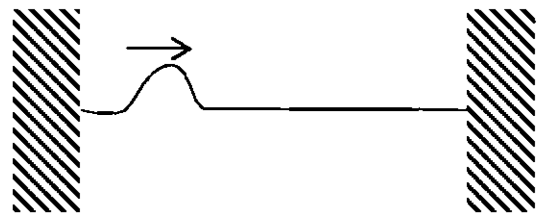
$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

## Standing Waves



$$\sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t)$$

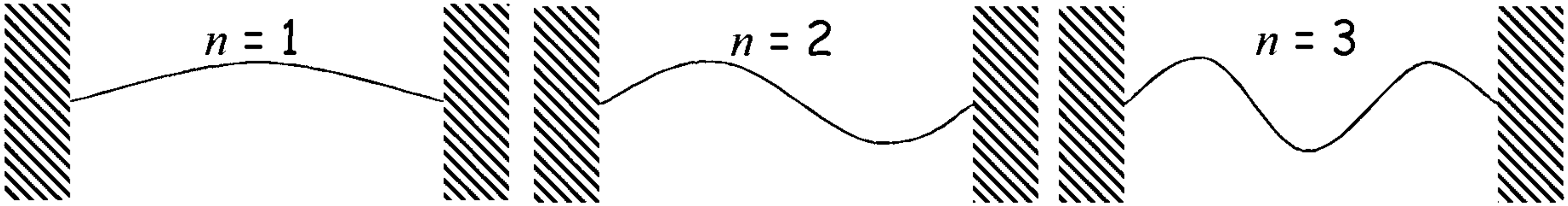
## Progressive Waves



$$f(x - vt)$$

# Fourier Analysis

Normal Modes: $y_n(x,t) = A_n \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t - \delta_n)$




*Any wave motion on the string can be described by a sum of these modes!*

Joseph Fourier, 1768-1830

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right)\cos(\omega_n t - \delta_n)$$

One equation, infinity unknowns ...

...thanks for nothin' Joe!

Focus on the shape first! $y(x)$

$$y(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

How to find a specific $B_n$? Multiply by $n^{th}$ harmonic and integrate.

This reduces series to 1 term!

Mathematically…

$$y(x) \qquad = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\sin\theta \sin\phi = \tfrac{1}{2}\left[\cos(\theta-\phi) - \cos(\theta+\phi)\right]$$

$$= \sum_{n=1}^{\infty} \frac{B_n}{2} \int_0^L \left[\cos\left(\frac{\left(n-n^*\right)\pi x}{L}\right) - \cos\left(\frac{\left(n+n^*\right)\pi x}{L}\right)\right] dx$$

$$= \sum_{n=1}^{\infty} \frac{B_n}{2} \left[ \frac{L}{(n-n^*)\pi} \sin\left( \frac{(n-n^*)\pi x}{L} \right) - \frac{L}{(n+n^*)\pi} \sin\left( \frac{(n+n^*)\pi x}{L} \right) \right]_0^L$$

$$= \sum_{n=1}^{\infty} \frac{B_n}{2} \left[ \underbrace{\frac{L}{(n-n^*)\pi}} \underbrace{\sin((n-n^*)\pi)} - \underbrace{\frac{L}{(n+n^*)\pi}} \underbrace{\sin((n+n^*)\pi)} - 0 + 0 \right]$$

$n \neq n^*$ $\quad = 0 \quad\quad = 0$ ..all terms zero!

$n = n^*$ $\quad = \frac{0}{0} \quad\quad = \frac{0}{\mathrm{x}}$ ...uh oh....

L'Hopital's Rule:

If $f(c)=0$ and $g(c)=0$ then $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$

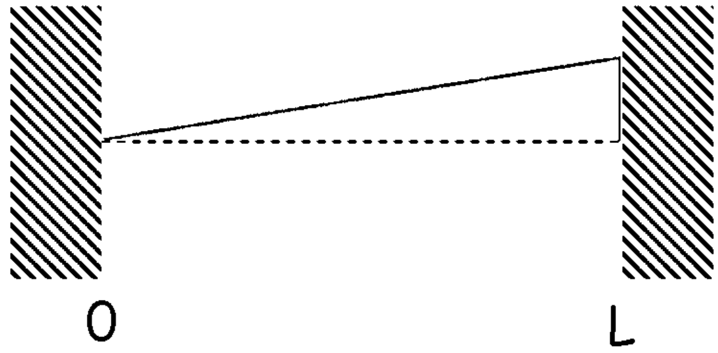
$$= \frac{B_{n^*} L}{2\pi} \frac{\pi \cos(0)}{1}$$

$$= \frac{B_{n^*} L}{2}$$

$$\int_0^L y(x) \sin\left(\frac{n^* \pi x}{L}\right) dx = \frac{B_{n^*} L}{2}$$

$$y(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \qquad B_n = \frac{2}{L} \int_0^L y(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Any shape *y(x)* between *0* and *L*




$$B_n = \frac{2}{L}\int_0^L cx \sin\left(\frac{n\pi x}{L}\right)dx \qquad c = \text{slope}$$

Integrate by parts:

$$\int_a^b f(x)g'(x)dx = \left[f(x)g(x)\right]_a^b - \int_a^b f'(x)g(x)dx$$

$$B_n = \frac{2c}{L}\left[-x\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Bigg|_0^L + \frac{L}{n\pi}\int_0^L \cos\left(\frac{n\pi x}{L}\right)\right]$$

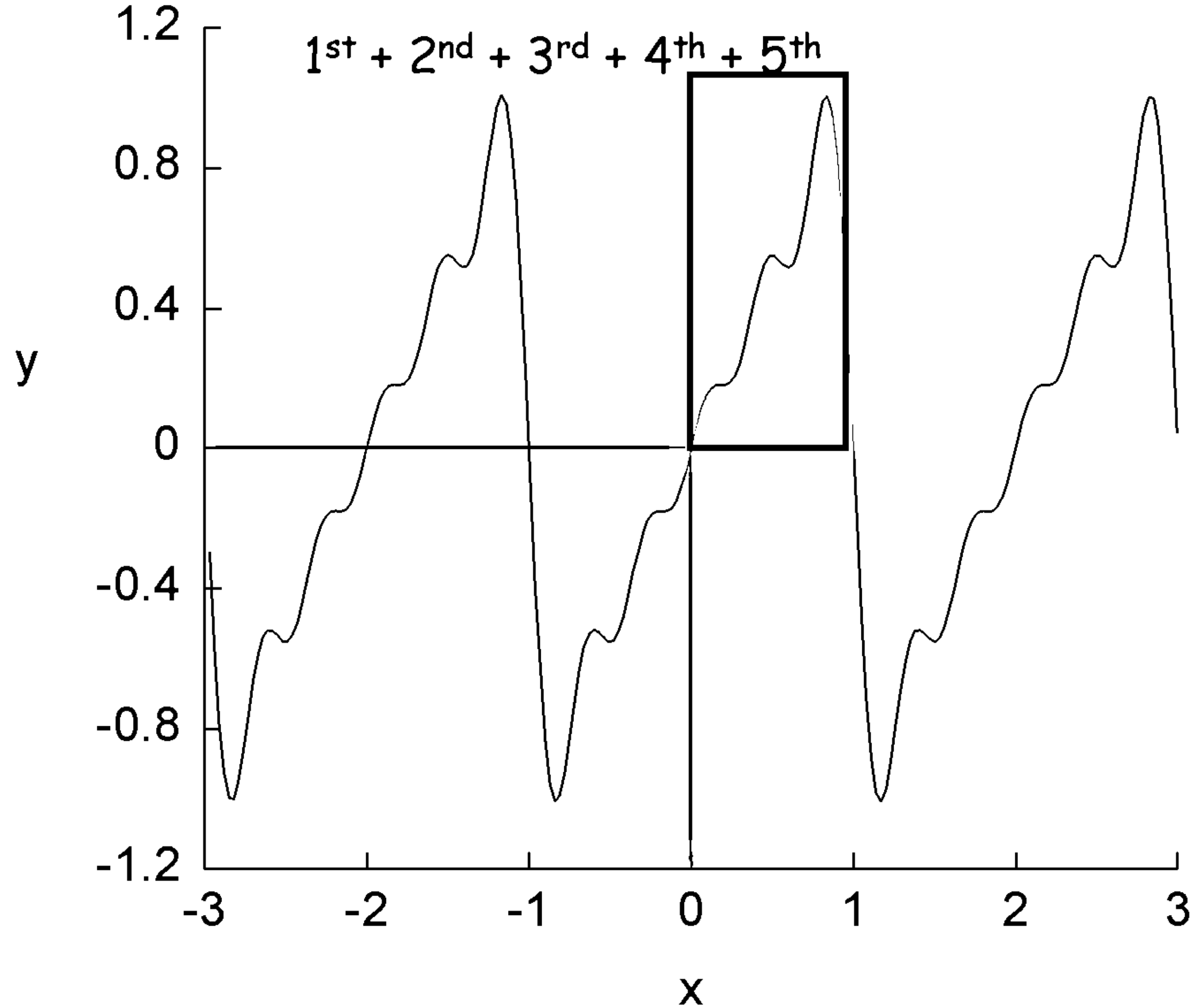
$$B_n = \frac{2c}{L}\left[ -x\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L + \frac{L}{n\pi}\int_0^L \cos\left(\frac{n\pi x}{L}\right)\right]$$

$$B_n = \frac{2c}{n\pi}\left[ -x\cos\left(\frac{n\pi x}{L}\right) + \frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\right]_0^L$$

$$B_n = \frac{2c}{n\pi}\left[ -L\cos(n\pi) + 0 + 0 - 0\right]$$

$$B_n = -\frac{2cL}{n\pi}\cos(n\pi)$$

$$y(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$



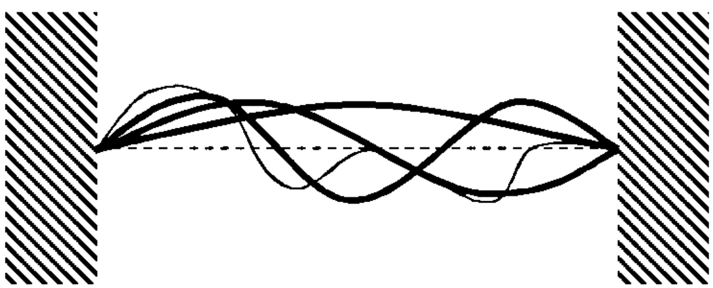
1st + 2nd + 3rd + 4th + 5th
1.2
0.8
0.4
0
-0.4
-0.8
-1.2
y
-3
-2
-1
0
1
2
3
x

# Les Foibles de Fourier en Frances

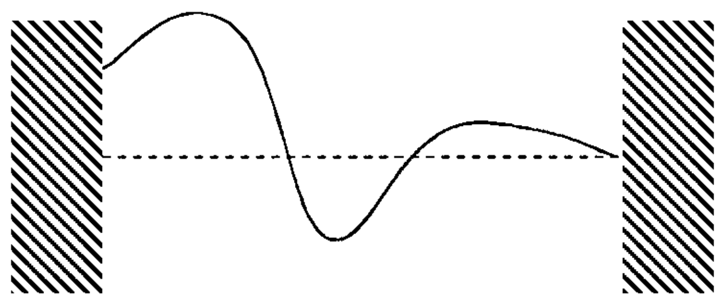
(French's Fourier Foibles)

$$y(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \qquad B_n = \frac{2}{L}\int_0^L y(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Good choice if boundaries are at 0:



Not so good for other shapes..



More General: function periodic on interval $x = -L$ to $x = L$.

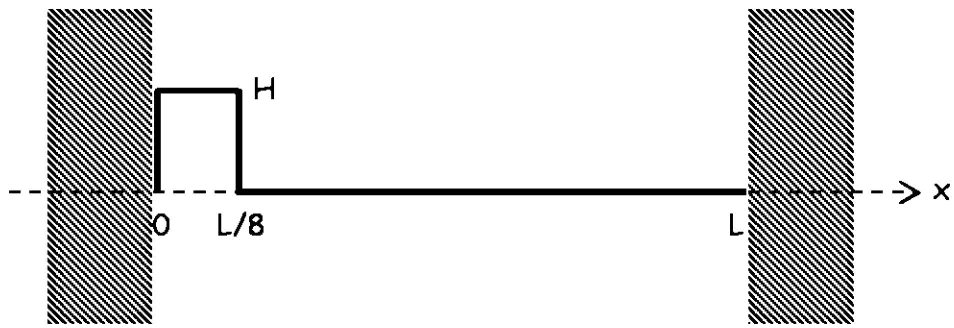
$$y(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$A_0 = \frac{1}{L}\int_{-L}^{L} y(x)\,dx$$

$$A_n = \frac{1}{L}\int_{-L}^{L} y(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$B_n = \frac{1}{L}\int_{-L}^{L} y(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

Something hard:




$$A_0 = \frac{1}{L}\int_0^{L/8} H dx + \frac{1}{L}\int_{L/8}^{L} 0 dx$$

$$\boxed{A_0 = \frac{H}{8}}$$

$$A_n = \frac{1}{L}\int_0^{L/8} H \cos\left(\frac{n\pi x}{L}\right) dx + \frac{1}{L}\int_{L/8}^{L} 0 \cos\left(\frac{n\pi x}{L}\right) dx$$

$$A_n = \frac{H}{L}\left(\frac{L}{n\pi}\right)\sin\left(\frac{n\pi x}{L}\right)\Bigg|_0^{L/8}$$

$$\boxed{A_n = \frac{H}{n\pi}\sin\left(\frac{n\pi}{8}\right)}$$

$$B_n = \frac{1}{L}\int_0^{L/8} H \sin\left(\frac{n\pi x}{L}\right)dx + 0$$

$$B_n = -\frac{H}{L}\left(\frac{L}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right)\Bigg|_0^{L/8}$$

$$\boxed{B_n = \frac{H}{n\pi}\left(1-\cos\left(\frac{n\pi}{8}\right)\right)}$$

$$y(x) = \frac{H}{8} + \sum_{n=1}^{\infty}\frac{H}{n\pi}\sin\left(\frac{n\pi}{8}\right)\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty}\frac{H}{n\pi}\left(1-\cos\left(\frac{n\pi}{8}\right)\right)\sin\left(\frac{n\pi x}{L}\right)$$

Any well behaved repetitive function can be described as an infinite sum of sinusoids with variable amplitudes (a Fourier Series). On a stretched string these correspond to the normal modes. Fourier analysis can describe arbitrary string shapes as well as progressive waves and pulses.