Superposition

Free SHM

EOM: $-kx = m\ddot{x}$

Solutions:

$x_1(t) = A_1 \cos(\omega_1 t + \phi_1)$

$x_2(t) = A_2 \cos(\omega_2 t + \phi_2)$

Superposition: the sum of solutions to an EOM is also a solution . . .

. . . if the EOM is linear.

$x$ and its derivatives appear only to first power.

... or a linear combination:

$c_1x_1 + c_2x_2$
Is the linear combination useful?

initial displacement, begin at rest
\( x_o, v_o = 0 \)

\[ x_1(0) = A_1 \cos(\phi_1) = x_0 \]

\[ \dot{x}_1(0) = -A_1 \omega_o \sin(\phi_1) = 0 \]

\[ \phi_1 = 0, \quad x_0 = A_1 \]

\[ x_1(t) = x_o \cos(\omega_o t) \]

initial velocity, begin at origin
\( v_o, x_o = 0 \)

\[ x_2(0) = A_2 \cos(\phi_2) = 0 \]

\[ \phi_2 = \frac{\pi}{2} \]

\[ \dot{x}_2(0) = -A_2 \omega_o \sin\left(\frac{\pi}{2}\right) = v_o \]

\[ A_2 = \frac{-v_o}{\omega_o} \]

\[ x_2(t) = \frac{-v_o}{\omega_o} \cos(\omega_o t + \frac{\pi}{2}) \]
initial displacement $x_o$ and velocity $v_o$

$$x_3(0) = A_3 \cos(\phi_3) = x_o$$

$$\dot{x}_3(0) = -A_3 \omega_o \sin(\phi_3) = v_o$$

Solve each for $A_3$, equate, solve for $\phi$:

$$\phi_3 = \tan^{-1} \left[ \frac{-v_o}{\omega_o x_o} \right]$$

Plug back into top equation to get $A_3$:

$$A_3 = \frac{x_o}{\cos(\tan^{-1}(-v_o/\omega_o x_o))}$$

Superposition: The motion resulting from two simultaneous initial conditions is equal to the sum of the motions resulting from each initial condition. . . . . . . if the EOM is linear.

Solution:

$$x_3(t) = \frac{x_o}{\cos(\tan^{-1}(-v_o/\omega_o x_o))} \cos(\omega_o t + \tan^{-1}(-v_o/\omega_o x_o))$$
The trig is getting complicated, let’s try something else...

Imagine SHM in 1D...

...as a projection of 2D motion.
Describe the position...

...geometrically \ OR \     ...algebraically
\[ \vec{r} = a\hat{i} + b\hat{j} \]
\[ z = a + jb \]

\( j \rightarrow "rotate \ 90 \ degrees" \)

\( j^2b \rightarrow "go \ along \ original \ axis \ in \ opposite \ direction" \)

\[ j^2 = -1 \]

\[ j = \sqrt{-1} \quad \text{... an imaginary number!} \]
\[
\sin(t) = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \frac{t^9}{9!} \ldots \\
\cos(t) = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!} \ldots 
\]

Circular motion in complex plane:

\[
\cos(t) + j\sin(t) = 1 + jt - \frac{t^2}{2!} - j\frac{t^3}{3!} + \frac{t^4}{4!} + j\frac{t^5}{5!} \ldots 
\]

\[
= 1 + jt + \frac{(jt)^2}{2!} + \frac{(jt)^3}{3!} + \frac{(jt)^4}{4!} + \frac{(jt)^5}{5!} \ldots 
\]

...expansion of \(\exp(jt)\)!

\[
\cos(t) + j\sin(t) = e^{jt} 
\]

“Euler’s Formula” - imaginary exponentials oscillate!
Describe SHM in the complex plane...

\[ z(t) = Ae^{j(\omega t + \phi)} \]

Does it work?

\[ -k[Ae^{j(\omega t + \phi)}] = m \frac{d^2}{dt^2}[Ae^{j(\omega t + \phi)}] \]

\[ -kAe^{j(\omega t + \phi)} = -mA\omega^2 e^{j(\omega t + \phi)} \]

Yes, if:

\[ k = m\omega^2 \]

\[ A \text{ and } \phi \text{ take any value} \]

Don't forget!!! \[ x(t) = \text{Re}[z(t)] = A \cos(\omega t + \phi) \quad \text{Keepin' it real!} \]
Same $A$ and $\omega$, but different $\phi$:

\[ z_1 = Ae^{j(\omega t + \phi_1)} \quad z_2 = Ae^{j(\omega t + \phi_2)} \]

\[ z_{1+2} = A\left(e^{j(\omega t + \phi_1)} + e^{j(\omega t + \phi_2)}\right) \]

\[ z_{1+2} = Ae^{j\omega t}\left(e^{j\phi_1} + e^{j\phi_2}\right) \frac{e^{j\phi_1}}{e^{j\phi_1}} \]

\[ z_{1+2} = Ae^{j(\omega t + \phi_1)}\left(1 + e^{j(\phi_2 - \phi_1)}\right) \]
\( x_1 \) and \( x_2 \) in phase

\[
\phi_1 - \phi_2 = 0
\]

\[
z_{1+2} = 2A e^{j(\omega t + \phi_1)}
\]

\[
x_{1+2} = 2A \cos(\omega t + \phi_1)
\]

\( x_1 \) and \( x_2 \) out of phase

\[
\phi_1 - \phi_2 = \pi
\]

\[
z_{1+2} = A e^{j(\omega t + \phi_1)}(1 + e^{j\pi})
\]

\[
z_{1+2} = 0
\]

\[
x_{1+2} = 0
\]
Same $A$ and $\phi$, different $\omega$:

Stick with trig...  
$$x_{1+2} = A[\cos(\omega_1 t) + \cos(\omega_2 t)]$$

$$\cos(\alpha) + \cos(\beta) = 2 \cos\left(\frac{\alpha+\beta}{2}\right) \cos\left(\frac{\alpha-\beta}{2}\right)$$

Then:  
$$x_{1+2} = 2A \cos\left(\frac{\omega_1 + \omega_2}{2} t\right) \cos\left(\frac{\omega_1 - \omega_2}{2} t\right)$$

$\omega_1 \approx \omega_2$ yields beats.
Systems with linear EOMs obey the Principle of Superposition: solutions to the EOM can be summed to make more complicated solutions.